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Quantum Logics and Convex Spaces

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An orthomodular σ-lattice with rich set of states satisfying the property that every affine functional from the set of states into the unit interval of the reals corresponds to an expectational functional of exactly one real observable (so-called u-spectral logic) is compared with the noncommutative spectral theory of Alfsen and Shultz. Necessary and sufficient conditions are found under which these two approaches are in correspondence.

INTRODUCTION

In the present paper, relations between orthomodular lattices and convex spaces are investigated. Problems of this kind arise in attempts to find a suitable mathematical foundation of quantum mechanics. In particular, the following two approaches are of our main interest: the quantum logic approach and the convexity approach.

The quantum logic approach was initiated by the well-known paper by Birkhoff and von Neumann (1936), who considered the event structure of a quantum mechanical system (the "quantum logic") as a continuous geometry, instead of a Boolean algebra describing the event structure of a classical system. In a more recent development, the main tool describing the quantum mechanical events has become a (σ-complete) orthomodular lattice, or, more generally, a (σ-complete) orthomodular poset. The main difference between classical and quantum event structures is that the latter need not be distributive, which reflects the fact that there exist events that cannot be measured simultaneously, as manifested by the Heisenberg uncertainty relations. An axiomatic model based on probabilistic ideas has been suggested by Mackey (1963). Taking as basic concepts states and observables of a physical system, and

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postulating the existence of a probability distribution over Borel subsets of a real line for a given observable in a given state, subjected to a few natural axioms, one arrives at an orthomodular poset describing the event structure, on which the (convex) set of states (generalized probability measures) is order determining. In this setting, observables (physical quantities) are described by σ-homomorphisms from Borel subsets of a real line to the quantum logic. Addition of observables is not defined unless the observables are compatible (i.e., their ranges belong to the same Boolean subalgebra of the quantum logic, such observables represent simultaneously measurable physical quantities). Gudder (1965) proved that if the existence of sums of bounded observables is postulated, where the addition is defined via linearity of expectational functionals (similar to the Segal or C*-algebraic approach), the quantum logic has a lattice structure. The existence and uniqueness of sums of bounded observables requires that observables are uniquely determined by their expectation values over a sufficiently large set of states. This property is nontrivial; there are examples of quantum logics with ambient sets of states in which observables are not uniquely determined by their expectations (Navara, 1995).

In the convex, or operational, approach to quantum mechanics, the starting point is the convex set of states of a physical system. The convexity assumption seems to be quite natural and corresponds to probablistic mixtures. The convex set of states is used as a cone base of an ordered linear space. The main tools then are a base norm space and an order unit space in separating order and norm duality, which are supposed to represent the space of (signed) measures and the space of observables. A connection with the quantum logic approach has been shown by Alfsen and Shultz (1976), who proved that if the corresponding spaces are in so-called spectral duality, then the extreme points in the unit interval of the order unit space form an orthomodular σ-lattice, and all other elements of this space can be considered as the usual bounded observables on it.

In the present paper, we start with so-called u-spectral logic, that is, an orthomodular σ-lattice with a rich state space, and assume that every convexity-preserving mapping from the state space to the real unit interval coincides to the expectation functional of exactly one observable. We prove that from this unique assumption, which also has a clear physical interpretation, a linear structure of bonded observables can be derived. Moreover, the set of observables admits the structure of an order unit space, and we arrive at an order unit and a base norm space derived from the state space which are in separating order and norm duality. With the additional assumption of the distributivity of the Segal product, we arrive at a Jordan algebra structure of bounded observables.

In the next part of the paper, we give a brief review of the Alfsen and Shultz theory, and we find a necessary and sufficient condition under which an order unit and base norm space in spectral duality give rise to a u-spectral logic.

Then we return to u-spectral logics and show that the corresponding order unit and base norm spaces are in spectral duality if and only if a kind of conditioning of states can be defined.

From the huge literature devoted to this subject, we cite only that which is most useful to us. Our main sources are as follows: for the quantum logic approach and orthomodular structures, Beltrametti and Cassinelli (1981), Pták and Pulmannová (1991), and Varadarajan (1985); for the convexity approach, Alfsen and Shultz (1976), Alfsen *et al.* (1978), Iochum (1984), and references cited therein; for order unit and base norm spaces, Alfsen (1971), and Asimov and Ellis (1980); for sum logics, Gudder (1965), Hudson and Pulmannová (1993); for u-spectral logics, Rüttimann (1981), and Pták and Pulmannová (1991); for conditioning of states, Pool (1968a,b), Guz (1980), and Edwards and Rüttimann (1990).

1. QUANTUM LOGICS, BASIC DEFINITIONS, AND RESULTS

A quantum logic (or more briefly a logic) $(L, \leq, ', 0, 1)$ is a σ-complete orthomodular poset. Thus, L is a partially ordered set with first and last elements 0 and 1, respectively, and with an orthocomplementation $': L \to L$ such that (i) $a \leq b \Rightarrow b' \leq a'$, (ii) $a'' = a$, (iii) $a \vee a' = 1$, (iv) for every sequence $(a_i)_{i\in\mathbb{N}} \subset L$ such that $a_i \leq a_j'$, $i \neq j$, $\vee_{i\in\mathbb{N}} a_i$ exits in L, and (v) $a \leq b \Rightarrow b = a \vee (a' \wedge b)$. Here the symbols $\vee$ and $\wedge$ denote the supremum and infimum in L if they exist. The property (v) is the orthomodular law. If L is a lattice, then L is an orthomodular σ-lattice (σ-OML). Two elements a, b in L are *orthogonal* (written $a \perp b$) if $a \leq b'$, and two elements a, b in L are *compatible* if there are three pairwise orthogonal elements a_1, b_1, c such that $a = a_1 \vee c$ and $b = b_1 \vee c$.

A mapping $\Phi: L_1 \to L_2$ is called a σ-*homomorphism* if (i) $a \perp b \Rightarrow \Phi(a) \perp \Phi(b)$, (ii) $\Phi(1) = 1$, (iii) $(a_i)_{i\in\mathbb{N}}$, $a_i \perp a_j$ whenever $i \neq j \Rightarrow \Phi(\vee_{i\in\mathbb{N}} a_i) = \vee_{i\in\mathbb{N}}\Phi(a_i)$. If, in addition, $\Phi(a \vee b) = \Phi(a) \vee \Phi(b)$ [dually $\Phi(a \wedge b) = \Phi(a) \wedge \Phi(b)$], whenever $a \vee b$ (dually $a \wedge b$) exists in L, then we call Φ a lattice σ-homomorphism. It turns out that every σ-homomorphism from a Boolean σ-algebra into a logic is a lattice σ-homomorphism.

A (*positive*) *measure* on L is a mapping $m: L \to R^+$ such that $m(a \vee b) = m(a) + m(b)$ whenever a and b are orthogonal. A measure m is

- —σ-*additive* if $(a_i)_{i\in\mathbb{N}} \subset L$, $a_i \perp a_j$, $i \neq j \Rightarrow m(\vee_i a_i) = \Sigma_i m(a_i)$
- —*completely additive* if $m(\vee_i a_i) = \Sigma_i m(a_i)$ for any subset (a_i) of pairwise orthogonal elements of L such that the supremum $\vee_i a_i$ exists
- —a *state* if $m(1) = 1$

It is easy to see that set of all (σ-additive) states of L is σ-convex, i.e., for any positive countable partition of unity of $\mathbb{R}$ $(\alpha_i)_{i\in\mathbb{N}}$ and any countable set $(m_i)_{i\in\mathbb{N}}$ of states, the mapping $a \to \Sigma_i\, \alpha_i m_i(a)$ is a (σ-additive) state on L.

We say that L admits a rich set M of (σ-additive) states if

$$a, b \in L, a \not\leq b \Rightarrow \exists m \in M: \quad m(a) = 1, \quad m(b) \neq 1$$

In particular, for every nonzero $a \in L$ there is a state $m \in M$ with $m(a) = 1$, that is, a rich set M is *unital*.

Let us consider a quantum logic L with a nonempty convex set of (σ-additive) states M. For every $m \in M$ and $\alpha \in \mathbb{R}^+$, denote by αm the mapping αm: $L \to \mathbb{R}^+$, $(\alpha m)(a) = \alpha m(a)$. Clearly, αm is a σ-additive measure on L. Denote $V^+(M) = \{\alpha m: \alpha \in [0, \infty), m \in M\}$, $V(M) = V^+(M) - V^+(M)$, $V_1^+(M) = \{\alpha m: \alpha \in [0, 1], m \in M\}$. It is easy to see that $V(M)$ is an ordered linear space with a generating cone $V^+(M)$ and M is a convex cone base (Rüttimann, 1981). A state $m \in M$ is called *pure* if it is an extreme point in M.

Let L be any logic and $(\Omega, \mathcal{S})$ be a measurable space. An *observable* is a σ-homomorphism x: $\mathcal{S} \to L$. If $(\Omega, \mathcal{S}) \equiv (\mathbb{R}, \mathcal{B}(\mathbb{R}))$, where $\mathcal{B}(\mathbb{R})$ denotes the Borel sets, we obtain a *real* observable. If x is an observable and m is σ-additive state on L, then $m \circ x$: $\mathcal{S} \to [0, 1]$ is a probability measre on $\mathcal{S}$, and the *expectation* of x in m is given by

$$m(x) = \int_\Omega \omega m(x(d\omega))$$

if the integral exists. Moreover, if f: $\Omega \to \mathbb{R}$ is a $(\mathcal{S}, \mathcal{B}(\mathbb{R}))$-measurable function, then $f(x) = x \circ f^{-1}$ is a (real) observable on L (called the function f of x), and

$$m(f(x)) = \int_{\mathbb{R}} m(x \circ f^{-1}(dt))$$

if the integral exists.

A subset K of L is a *sublogic* of L if (i) $0 \in K$, (ii) $a \in K \Rightarrow a' \in K$, (iii) $a_i \perp a_j (i \neq j)$ $(a_i)_{i\in\mathbb{N}} \subset K \Rightarrow \vee_{i\in\mathbb{N}}\, a_i \in K$. If, in addition, a, $b \in K$, $a \vee b$ exists in $L \Rightarrow a \vee b \in K$, then K is called a *lattice-sublogic* of L. A lattice-sublogic B such that B is a Boolean σ-algebra with the partial order and orthocomplementation inherited from L is a *Boolean sublogic* (or a Boolean sub-σ-algebra) of L. Two elements a, $b \in L$ are compatible iff they are contained in a Boolean sublogic of L [equivalently, if $a = (a \wedge b) \vee (a \wedge b')$, where the corresponding suprema and infima exists; see, e.g., Varadarajan (1985) and Pták and Pulmannová (1991)]. We write aCb if a and b are compatible. For any subset K of L, the set $C(K) = \{a \in L: aCk\ \forall k \in K\}$

is the *commutant of K*. For any $K \subset L$, $C(K)$ is a lattice sublogic of L. In particular, $C(L)$, the set of all "absolutely compatible" elements of L, is the *center of L*. Equivalently, $C(L)$ can be defined as the intersection of all maximal Boolean sublogics (so-called *blocks* of L), hence $C(L)$ is a Boolean sub-σ-algebra of L. Alternatively, $C(L)$ can be defined as the set of all elements a of L such that L can be decomposed into the direct sum of two intervals $L \equiv [0, a] \oplus [0, a']$. A logic L is called *irreducible* if its center is trivial, i.e., $C(L) = \{0, 1\}$. The set $C(C(K))$ is so-called *bicommutant* of a subset K of L. If L is a lattice and K consists of mutually compatible elements, then $C(C(K))$ is a Boolean sublogic of L (see, e.g., Varadarajan, 1985; Pták and Pulmannová, 1991).

Observables $x_1, \ldots, x_n$ from $(\Omega, \mathcal{S})$ to L are *compatible* if their ranges are contained in the same block of L. There is a functional calculus for compatible observables (Varadarajan, 1985), namely if $x_1, \ldots, x_n$ are compatible observables, then there is an observable u and measurable functions $f_1, \ldots, f_n$ such that $x_i = u \circ f_i^{-1} =: f_i(u)$, $i = 1, \ldots, n$. In particular, if x, y are compatible real observables, then the observables, $r.x$ ($r \in \mathbb{R}$), x^2, $x + y$, $|x|$, exp $(x + y)$, etc., can be defined. For example, the observable $x + y$ is defined as $u \circ (f + g)^{-1} = (f + g)(u)$ when $x = u \circ f^{-1} = f(u)$, $y = u \circ g^{-1} = g(u)$. If m is any σ-additive state on L, then

$$
\begin{aligned}
m(x + y) &= \int tm((x + y)(dt)) \\
&= \int tm(u \circ (f + g)^{-1} (dt) \\
&= \int (f + g)(t)m(u(dt)) \\
&= \int f(t)m(u(dt)) + \int g(t)m(u(dt)) \\
&= \int tm(u \circ f^{-1} (dt)) + \int tm(u \circ g^{-1} (dt)) \\
&= m(x) + m(y)
\end{aligned}
$$

Every real observable x has a *spectrum* i.e., a smallest closed subset C of $\mathbb{R}$ such that $x(C) = 1$. We will denote by $sp(x)$ the spectrum of x. An observable is *bounded* if its spectrum is a compact subset of $\mathbb{R}$. If $sp(x)$ is a subset of $\{0, 1\}$, we say that x is a *proposition observable.* To every $a \in L$, there is a proposition observable q_a such that $q_a\{1\} = a$. Let $\mathbb{O}(L)$ denote the set of all bounded observables on L. We may assume $L \subset \mathcal{O}(L)$. In particular, q_1, an observable with spectrum $\{1\}$, can be identified with the element 1 in L, and q_0, an observable with spectrum $\{0\}$, can be identified with the element 0 in L. An observable is *simple* if its spectrum is finite.

In what follows, we consider a logic L with a rich σ-convex set of σ-additive states M.

Lemma 1.1 (Gudder, 1965). A real observable x belongs to $\mathcal{O}(L)$ if and only if $|m(x)| < \infty$ for all $m \in M$.

Proof. If $x \in \mathcal{O}(L)$, then $|m(x)| < \infty$ for all $m \in M$. To prove the converse, assume that x is not bounded, and let $r_n \in sp(x)$ be such that $|r_n| > 2^{n+1}$ ($n \in \mathbb{N}$). Let $U_n = (r_n - \varepsilon, r_n + \varepsilon)$, $\varepsilon < 1/2$, be disjoint open intervals in $\mathbb{R}$. Put $a_n = x(U_n)$. Clearly, $a_n \neq 0$, and let $m_n \in M$ be such that $m_n(a_n) = 1$. Since a_n are mutually orthogonal, we have $m_i(a_j) = 0$ whenever $i \neq j$. Convexity of M implies that the state $m = \Sigma 2^{-n} m_n$ belongs to M, and we have

$$m(|x|) = \int |t| m(x(dt)) = \sum_n 2^{-n} \int_{U_n} |t| m(x(dt))$$
$$\geq \sum_n 2^{-n}(2^{n+1} - 1/2) = \infty$$

We will say that a real observable x on L is *positive* if $m(x) \geq 0$ $\forall m \in M$. Let $\mathcal{O}^+(L)$ denote the set of all positive bounded observables, and we denote by $\mathcal{O}_1(L)$ the set of all $x \in \mathcal{O}^+(L)$ such that $m(x) \leq 1$ for all $m \in M$.

Lemma 1.2. Let L be a logic with a unital set M of states. The following conditions are equivalent for $x \in \mathcal{O}(L)$:

(1) x is positive
(2) $sp(x) \subset [0, \infty)$
(3) $x = y^2$ for some $y \in \mathcal{O}(L)$

Proof. (1) $\Rightarrow$ (2): Let $\lambda < 0$. By the definition of spectrum, if $\lambda \in sp(x)$, we have $x(\mathcal{U}(\lambda)) \neq 0$ for every open neighborhood of λ. Hence, for any $\varepsilon > 0$, there is $m \in M$ with $M(x(\lambda - \varepsilon, \lambda + \varepsilon)) = 1$. But then $m(x) \leq \lambda + \varepsilon$, and choosing ε such that $\lambda + \varepsilon < 0$, we get $m(x) < 0$, contradicting the assumption.

(2) $\Rightarrow$ (3) follows by taking $y = f(t)$, where $f(t) = \sqrt{t}$ on $sp(x)$.

(3) $\Rightarrow$ (1) is clear. ■

Lemma 1.3. Every observable $x \in \mathcal{O}(L)$ can be written in the form $x = x^+ - x^-$, where x^+ and x^- are compatible observables in $\mathcal{O}^+(L)$.

Proof. Define $f^+(t) = \max\{0, t\}$, $f^-(t) = -\min\{0, t\}$, $t \in \mathbb{R}$. Then $f^+ - f^- = id$, the identity function on $\mathbb{R}$, and applying the functional calculus, we get $x = x \circ id^{-1} = x \circ (f^+ - f^-)^{-1} = x \circ (f^+)^{-1} - x \circ (f^-)^{-1}$. Putting $x^+ = x \circ (f^+)^{-1}$, $x^- = x \circ (f^-)^{-1}$, we obtain the desired decomposition. ■

Lemma 1.4. Let L be a logic with a unital set M of σ-additive states. Then for any $x \in \mathcal{O}(L)$,

$$\sup\{|m(x)|: m \in M\} = \max\{|t|: t \in sp(x)\}$$

Proof. For $m \in M$ we have $m(x) = \int_{sp(x)} tm(x(dt))$. Since $m(x(.))$ is a probability measure on $\mathcal{B}(\mathbb{R})$, we may conclude that $\inf sp(x) \leq m(x) \leq \max sp(x)$. Let $\delta = \max\{|t|: t \in sp(x)\}$; then $|m(x)| \leq \delta$ for all $m \in M$.

Without any loss of generality, we may assume that $\delta = \max sp(x)$. For any $\varepsilon > 0$ we have $a = x([\delta - \varepsilon, \delta]) \neq 0$. Unitality of M implies that there is $m \in M$ with $m(a) = 1$, so that $m(x) = \int_{[\delta-\varepsilon,\delta]} tm(x(dt)) > \delta - \varepsilon$. Hence $\delta = \sup\{|m(x)|: m \in M\}$. ■

We denote

$$\|x\|_1 = \sup\{|m(x)|: m \in M\} \tag{1.1}$$

and

$$\|x\|_2 = \max\{|t|: t \in sp(x)\} \tag{1.2}$$

By Lemma 1.4, $\|x\|_1 = \|x\|_2 =: \|x\|$. In the sequel we will prove that, under certain conditions, $\|x\|$ is a norm on $\mathcal{O}(L)$. Now we can only prove the following statement.

Lemma 1.5. On every maximal compatible subset of $\mathcal{O}(L)$, the function $x \to \|x\|$ is a norm. Moreover, the set of simple observables is $\|.\|$-dense in $\mathcal{O}(L)$.

Proof. If x, y, z are compatible observables, then they are Borel functions of one observable u, say. This implies that $x + y$ is compatible with z. Moreover, for every $m \in M$, $m(x + y) = m(x) + m(y)$. Therefore, a maximal subset of pairwise compatible observables is a real linear space and, using the definition of $\|.\|$ by (1.1), we prove that $\|x + y\| \leq \|x\| + \|y\|$. If $\|x\| = 0$, then by the definition of $\|.\|$ by (1.2), $sp(x) = \{0\}$, hence $x = q_0$. Clearly, $\|x\| \geq 0$ and $\|\alpha x\| = |\alpha|\|x\|$ for any real α. It follows that $x \to \|x\|$ is a norm.

For every $x \in \mathcal{O}(L)$, we can write $x = id_{\mathbb{R}}(x)$, $id_{\mathbb{R}}(t) = t$, $t, \in \mathbb{R}$. Assume that $x \in \mathcal{O}^+(L)$; then there is a nondecreasing sequence of simple nonnegative functions f_n such that $f_n \to id_{\mathbb{R}}$ in the supremum norm of functions. This implies that $\|f_n(x) - x\| \to 0$, where $(f_n(x))$ are simple observables. [See, e.g., Pták and Pulmannová (1991), Chapter 4, for the details.]

Let $x \in \mathcal{O}(L)$ be arbitrary. Taking into account that every $x \in \mathcal{O}(L)$ can be decomposed into positive and negative parts, we can find a sequence of simple functions g_n such that $\|g_n(x) - x\| \to 0$. ■

For any $x \in \mathcal{O}(L)$, the mapping $m \mapsto m(x)$ is a linear functional on $V(M)$. Denote for $v \in V(M)$,

$$\|v\| = \sup\{|v(x)|: x \in \mathcal{O}_1(L)\}$$

It is easy to see that $\|.\|$ is a norm on $V(M)$ and, moreover, $\|m\| = m(1) = 1$ for all $m \in M$.

A *weak topology* on $V(M)$ will be defined by the base of neighborhoods

$$U(v_0, x_1, \ldots, x_n, \varepsilon) = \{v \in V(M): |v(x_i) - v_0(x_i)| \leq \varepsilon, i = 1, 2, \ldots, n\}$$

where $v_0, v \in V(M)$, $x \in \mathcal{O}_1(L)$, $\varepsilon > 0$. Clearly, a net $\{v_\alpha\}$ in $V(M)$ weakly converges to $v \in V(M)$ if and only if $v_\alpha(x) \to v(x)$ $\forall x \in \mathcal{O}(L)$.

A linear transformation T: $V(M) \to V(M)$ is *weakly continuous* if $v_\alpha \to v \Rightarrow Tv_\alpha \to Tv$. Moreover, T is called *positive* if $T(V^+(M)) \subset V^+(M)$.

Lemma 1.6. A net $\{v_\alpha\} \subset V(M)$ weakly converges to $v \in V(M)$ if and only if $v_\alpha(a) \to v(a)$ for all $a \in L$.

Proof. Necessity is clear. To prove sufficiency, assume that $v_\alpha(a) \to v(a)$ for any $a \in L$. For any simple observable $x \in \mathcal{O}(L)$, $x = \Sigma_{i=1}^n t_i q_{a_i}$, where a_i, $i = 1, \ldots, n$ are mutually orthogonal elements of L, we then have $v_\alpha(x) \to v(x)$. Since, by Lemma 1.2, the set of simple observables is $\|.\|$-dense in $\mathcal{O}(L)$, the result follows. ■

The set M induces a weak topology on the set $\mathcal{O}(L)$ in the following way. For any $x_1 \in \mathcal{O}(L)$, the base of open neighborhoods is formed by finite intersections of the sets $\mathcal{U}(x_1, m, \varepsilon) = \{x \in \mathcal{O}(L): |m(x_1) - m(x)| \leq \varepsilon\}$.

Accordingly, a linear transformation T: $\mathcal{O}(L) \to \mathcal{O}(L)$ is *weakly continuous* if $x_\alpha \to x \Rightarrow Tx_\alpha \to Tx$ for any net (x_α) weakly convering to x, and T is *positive* if $T(\mathcal{O}^+(L)) \subset \mathcal{O}^+(L)$. A positive linear transformation T of $\mathcal{O}(L)$ is called *normal* if $x_\alpha \uparrow x \Rightarrow Tx_\alpha \uparrow Tx$, where $x_\alpha \uparrow x$ means that the net x_α is nondecreasing and has x for supremum. Clearly, a positive, weakly continuous linear transformation T is normal.

In the special case when the set M coincides with the set of all σ-additive states on L, we have the following characterization of normal positive transformations of $\mathcal{O}(L)$.

Proposition 1.7. Let M coincide with the set of all σ-additive states on L. Then a positive transformation T of $\mathcal{O}(L)$ is normal if and only if

(1) $T1 = 1$

(2) $T(\vee a_i) = \Sigma\, Ta_i$ for any sequence $(a_i)_i$ of mutually orthogonal elements of L.

Proof. Necessity easily follows from the fact that (identifying a_i with q_{a_i}), $\vee_{i=1}^n a_i = \Sigma_{i=1}^n q_{a_i} = \Sigma_{i=1}^n a_i$ $\forall n \in \mathbb{N}$. To prove sufficiency, assume that

T satisfies the latter properties. Then we have $T(\mathcal{O}_1(L)) \subset \mathcal{O}_1(L)$. Define, for any $m \in M$, $T^{\#}(m)$: $L \to [0, 1]$ by $T^{\#}(m)(a) = m(Ta)$. For any sequence (a_i) of mutually orthogonal elements of L, $T^{\#}(m)(\vee a_i) = m\,(T(\vee a_i)) = m\,(\Sigma\, Ta_i) = m(\sup_n \Sigma_{i=1}^n Ta_i) = sup_n(\Sigma_{i=1}^n m(Ta_i)) = \Sigma\, m(Ta_i) = \Sigma\, T^{\#}(m)(a_i)$; hence $T^{\#}(m)$ is a σ-additive state on L. Now $x_\alpha \uparrow x$ means that $m(x_\alpha) \uparrow m(x)$ for all $m \in M$, hence $T^{\#}(m)(x_\alpha) \uparrow T^{\#}(m)(x)$ for all $m \in M$, and hence $m(T(x_\alpha)) \uparrow m(T(x))$ for all $m \in M$. ■

2. U-SPECTRAL LOGICS

A logic L with a rich σ-convex set of states M is called *u-spectral* if for every affine functional f: $M \to [0, 1]$ there is a unique real observable x such that $f(m) = m(x)$ for every $m \in M$.

Projection lattices of von Neumann algebras yield an example of u-spectral logics (Rüttimann, 1981).

In what follows, we assume that (L, M) is a u-spectral logic.

Lemma 2.1. If $x, y \in \mathcal{O}(L)$, and $m(x) = m(y)$ for all $m \in M$, then $x = y$.

Proof. Observe that for any observable x, $y = \|x\|q_1 + x \in \mathcal{O}^+(L)$, and $y/\|y\| \in \mathcal{O}_1(L)$. Therefore, $m \mapsto m(y/\|y\|)$ defines an affine functional on M with values in $[0, 1]$, hence the observable $y/\|y\|$, hence also y is uniquely defined by its expectation values $m(y)$, $m \in M$, and the same is true for $x = y - \|x\|q_1$. ■

Lemma 2.1 claims that u-spectral logic has the Uniqueness property (Gudder, 1966). In the next proposition we show that also the Existence property is satisfied. It follows that L is a lattice (Gudder, 1966).

Proposition 2.2. Let (L, M) be a u-spectral logic. The set $\mathcal{O}(L)$ of all bounded real observables can be equipped with a structure of a real linear space.

Proof. If x and y are compatible, then $x + y$ exists by the functional calculus, and we have $m(x + y) = m(x) + m(y)$ for all $m \in M$. In particular, $q_0 + x = x$ for any $x \in \mathcal{O}(L)$.

Assume first that $x, y \in \mathcal{O}_1(L)$. For any fixed $\alpha \in (0, 1)$, the mapping $\nu(m) = \alpha m(x) + (1 - \alpha)m(y)$ is an affine functional from M to $[0, 1]$. Therefore there is a unique observable $z \in \mathcal{O}_1(L)$ such that, for any $m \in M$,

$$m\,(x) = \alpha m(x) + (1 - \alpha)m(y)$$

We define $z = \alpha x + (1 - \alpha)y$.

Now let $x, y \in \mathcal{O}^+(L)$. Without loss of generality we may assume that $x \neq 0$ and $y \neq 0$. Then $x/\|x\| \in \mathcal{O}_1(L)$, $y/\|y\| \in \mathcal{O}_1(L)$. Choosing $\alpha = \|x\|/$

$(\|x\| + \|y\|)$, we obtain that $z = \alpha x/\|x\| + (1 - \alpha)y/\|y\|$ is defined, and put $x + y = (\|x\| + \|y\|)z$.

Now for any $x, y \in \mathcal{O}(L)$ the observable $z = (\|x\|q_1 + x) + (\|y\|q_1 + y)$ is defined, and also $w = z - (\|x\| + \|y\|)q_1$ is defined, and we put $w = x + y$. Clearly, $m(w) = m(x) + m(y)$ for all $m \in M$.

It is easy to check that $(\mathcal{O}(L), +)$ with the above-defined operation $+$, and with the real multiplication defined by functional calculus for observables, forms a real linear space. ■

Clearly, $\mathcal{O}^+(L)$ is a positive cone in $\mathcal{O}(L)$. Moreover, every observable can be uniquely decomposed in the form $x = x^+ - x^-$, where x^+ and x^-, the positive and negative parts of x, are positive observables defined by the functional calculus (Lemma 1.3). Therefore, $\mathcal{O}^+(L)$ is a generating cone for $\mathcal{O}(L)$, and so $\mathcal{O}(L)$ is a directed real linear space. The following proposition shows that it is a normed space as well.

Proposition 2.3. The function $x \mapsto \|x\|$ defined in two equivalent ways by (1.1) and (1.2) is a norm on $\mathcal{O}(L)$.

Proof. As a direct consequence of the definition, we obtain that $\|x\| \geq 0$, $\|x\| = 0 \Rightarrow x = q_0$, and $\|\alpha x\| = |\alpha|\|x\|$ for any real α. It remains to prove $\|x + y\| \leq \|x\| + \|y\|$. We have

$$\begin{aligned} \|x + y\| &= \sup\{|m(x + y)|: m \in M\} \\ &\leq \sup\{|m(x)| + |m(y)|: m \in M\} \\ &\leq \sup\{|m(x)|: m \in M\} + \sup\{|m(y)|: M \in M\} \\ &= \|x\| + \|y\| \quad ■ \end{aligned}$$

Lemma 2.4. The set $\mathcal{O}(L)$ is monotone complete, that is, if $\{x_\alpha\}$ is a nondecreasing net in $\mathcal{O}(L)$ bounded above, then it has a supremum $x \in \mathcal{O}(L)$. Moreover, x_α converges to x in weak topology.

Proof. Without loss of generality we may assume that $x_\alpha \geq 0$, and let $\|x_\alpha\| \leq K$, $K > 0$. For each $m \in M$, the net $\{m(x_\alpha)/K\}$ is nondecreasing and bounded above by 1, therefore it has a limit, $\nu(m)$, say. The mapping ν: $M \to [0, 1]$ is an affine functional, therefore there is $y \in \mathcal{O}_1(L)$ such that $\nu(m) = m(y)$. Writing $x = Ky$, we see that $\{x_\alpha\}$ converges to x in weak topology. It remains to prove that x is the supremum of $\{x_\alpha\}$. Clearly, $x_\alpha \leq x\ \forall\alpha$, and if $x_\alpha \leq z\ \forall\alpha$, then for any $m \in M$, $\lim_\alpha m(x_\alpha) = m(x) \leq m(z)$, hence $x \leq z$. ■

Proposition 2.5. The set $\mathcal{O}(L)$ is norm-complete.

Proof. Let $\{x_n\}_0^\infty$ be a Cauchy sequence from $\mathcal{O}(L)$, and assume without loss of generality that $\|x_n - x_{n-1}\| \leq 2^{-n}$ for $n = 1, 2, \ldots$. Writing

$$y_n = x_0 + \sum_{j=1}^{n} [(x_j - x_{j-1}) + 2^{-j}q_1] = x_n + (1 - 2^{-n})q_1$$

we get an increasing net $\{y_n\}_0^\infty$ such that $\|y_n - y_{n-1}\| \leq 2^{-n+1}$. Since $\|y_n\| \leq \|y_0\| + \|y_1 - y_0\| + \cdots + \|y_n - y_{n-1}\| \leq \|y_0\| + 1$ for all n, $\{y_n\}$ is bounded above, so by Lemma 2.4, it has a pointwise limit y in $\mathcal{O}(L)$. In fact, $\{y_n\}$ is norm-Cauchy, and $\|y - y_n\| \leq 2^{-n+1}$. Therefore

$$\|(y - q_1) - x_n\| = \|(y - y_n) - 2^{-n}q_1\| \leq 3.2^{-n}$$

Hence $\{x_n\}$ converges in norm to the limit $y - q_1$. ■

The set $\mathcal{O}_1(L) := \{x \in \mathcal{O}^+(L): \|x\| \leq 1\}$ with the partial operation $\oplus$ defined if and only if $x + y \in \mathcal{O}_1(L)$ and then $x \oplus y = x + y$ becomes an interval effect algebra with the ambient group $\mathcal{O}(L)$ (Foulis and Bennett, 1994). The following lemma shows that there is an analogy of a range projection in von Neumannn algebras for the elements in $\mathcal{O}_1(L)$.

Lemma 2.6. To every $y \in \mathcal{O}_1(L)$, there is a unique element $a \in L$ such that $m(y) = 0$ if and only if $m(a) = 0$.

Proof. We have, for any $m \in M$, $m(y) = 0$ iff $m(q_1 - y) = \int_0^1 tm((q_1 - y)(dt)) = 1$ iff $(q_1 - y)\{1\} = 1$ iff $(q_1 - y)\{1\}' = 0$. Putting $a = \int_0^1 (q_1 - y)(\{1\})'$, we obtain the existence statement; uniqueness is clear. ■

We also have $m(y) = 1$ iff $m(y\{1\}) = 1$.

According to Gudder (1996), the states in M have the Jauch–Piron property, i.e., $m(a) = 1$ and $m(b) = 1$ imply $m(a \wedge b) = 1$. We have the following equivalent characterzations of "sharp" elements of $\mathcal{O}_1(L)$.

Lemma 2.7. Let $x \in \mathcal{O}_1(L)$. The following statements are equivalent:

(i) $x \in \mathcal{O}_1(L)$ is a proposition observable

(ii) x is an extreme point in the convex set $\mathcal{O}_1(L)$

(iii) $x^2 = x$

(iv) $x \wedge (1 - x) = 0$

(v) $x \vee (1 - x) = 1$

Proof. (i) ⇒ (ii): Assume that $x = \alpha y + (1 - \alpha)z$ with $y, z \in \mathcal{O}_1(L)$. Clearly $m(x) = 0, 1$ iff $m(y) = m(z) = 0, 1$, respectively. Therefore, by the richness and the Jauch–Piron property of M, $x(\{1\}) = y(\{1\}) \wedge z(\{1\})$, $x(\{0\}) = y(\{0\}) \wedge z(\{0\})$. Since x is a proposition observable, we have

$$1 = x(\{0\}) \vee x(\{1\}) = y(\{0\}) \wedge z(\{0\}) \vee y(\{1\}) \wedge z(\{1\})$$
$$\leq y(\{0\}) \vee y(\{1\}), z(\{0\}), \vee z(\{1\})$$

whence y, z are also proposition observables. Moreover, from

$$x(\{1\}) = y(\{1\}) \wedge z(\{1\}) \leq y(\{1\}), z(\{1\})$$
$$\leq y(\{1\}) \vee z(\{1\}) = (y(\{0\}) \wedge z(\{0\}))' = x(\{0\})' = x(\{1\})$$

it follows that $x(\{1\}) = y(\{1\}) = z(\{1\})$, hence $x = y = z$. This means that x is an extreme point.

(ii) $\Rightarrow$ (iii) Let x be an extreme point of $\mathcal{O}_1(L)$ and consider the functions $\phi(t) = t^2$ and $\psi(t) = 2t - t^2$ defined for $t \in [0, 1]$. These functions satisfy $\frac{1}{2}\phi + \frac{1}{2}\psi = id$. Since $sp(x) \subset [0, 1]$, we can form functions $\phi(x)$ and $\psi(x)$ and, according to the functional calculus, we have $x = \frac{1}{2}\phi(x) + \frac{1}{2}\psi(x)$. Since x is an extreme point, we must have $\phi(x) = \psi(x)$. From this (iii) follows.

(iii) $\Rightarrow$ (iv) Let $x^2 = x$, and assume that there is $x \in \mathcal{O}_1(L)$ such that $z \leq x$ and $z \leq (1 - x)$. Using Lemma 2.6, we obtain

$$x\{1\} \leq (1 - x)\{0\} \leq z\{0\}, \qquad x\{0\} \leq z\{0\}$$

hence $1 = x\{1\} \vee x\{0\} \leq z\{0\}$, which implies that $z\{0\} = 1$, i.e., $z = q_0 = 0$.

(iv) $\equiv$ (v) holds by duality.

(iv) $\Rightarrow$ (iii) If $x \wedge (1 - x) = 0$, then $0 \leq x^2 \leq x \leq 1$ implies $x - x^2 = x(1 - x) \in \mathcal{O}_1(L)$, $x - x^2 \leq x$, and similary, $(1 - x) - (1 - x)^2 = x - x^2 \leq 1 - x$, which yields $x^2 = x$.

(iii) $\Rightarrow$ (i) By the continuity of $f(t) = t^2$, $sp(x^2) = sp(x)^2$, and from $x = x^2$ it follows that $sp(x) \subset \{0, 1\}$. ■

We recall (Alfsen, 1971; Asimov and Ellis, 1980) that an *order-unit space* is a partially ordered normed vector space with a distinguished order unit e which is Archimedean in that $na \leq e$ for $n = 1, 2, \ldots$ implies $a \leq 0$, and with the norm given by

$$\|a\| = \inf\{\lambda > 0: -\lambda e \leq a \leq \lambda e\} \tag{2.1}$$

A linear functional p on an order-unit space (A, e) is called a *state* if $p \geq 0$ and $p(e) = 1$. The states of (A, e) form a weakly compact convex subset of A^* which is called the *state space* of (A, e) and will be denoted by $S(A, e)$ or simply by S. The crucial property of order-unit spaces is that there exist sufficiently many states in the following sense:

For every $a \in A$, $a \geq 0$ iff $p(a) \geq 0$ for all $p \in S$ and, moreover, $\|a\| = \sup\{|p(a)|: p \in S\}$ (Alfsen and Shultz, 1976, Proposition II.1.7).

Recall that a convex subset K of a hyperplane H not passing through the origin of a vector space E is said to be a *base* for the cone $C = \cup_{\lambda \geq 0} \lambda K$. A convex subet B of a vector space E is *radially compact* if $B \cap L$ is a closed and bounded segment for every line L through the origin of E. We shall use the term *base-norm space* and the notation (E, K) to denote a directed vector space E for which E^+ has a base K such that B = conv $(K \cup -K)$ is radially compact, considered as a normed space in the *base-norm* defined by the Minkowski functional

$$\|x\| = \inf\{\lambda \geq 0: x \in \lambda B\}$$

If (E, K) is a base-norm space, then by definition there is a linear functional f on E such that $K \subset f^{-1}(1)$. By the directedness f is uniquely determined. It is called the linear functional that *carries* K, and will be denoted by e_K. The norm is additive on E^+; moreover, $\|x\| = e_K(x)$ for $x \in e^+$ (Alfsen and Shultz, 1976, Proposition II.1.13).

Recall that every element x of a base-norm space (E, K) admits a decomposition $x = y - z$, where $y, z \geq 0$ and $\|x\| = \|y\| + \|z\|$.

It is a well-known fact that the dual of an order unit space (A, e) is the base-norm space (A^*, S) and the dual of a base-norm space (E, K) is the order-unit space (E^*, e_K).

Theorem 2.8. Let (L, M) be a u-spectral logic. The set $\mathbb{O}(L)$ of bounded real observables endowed with the norm $\|x\| = \sup\{|m(x)|: m \in M\} = \sup\{|t|: t \in sp(x)\}$ and ordering defined by $x \geq 0$ iff $m(x) \geq 0$ for all $m \in M$ becomes a norm complete order unit space in which the order unit is the observable q_1 and the following condition is satisfied:

$$-1 \leq a \leq 1 \quad \text{implies} \quad 0 \leq a^2 \leq 1 \tag{2.2}$$

Proof. We follow the pattern from Hudson and Pulmannová (1993). Proposition 5.2.

Clearly, $nx \leq q_1$ for all $n \in \mathbb{N}$ iff $nm(x) \leq 1$ for all $n \in \mathbb{N}$, hence $m(x) = 0$ $\forall m \in M$, hence $x = q_0 = 0$. This shows the Archimedean property.

In a similar way, $-\lambda q_1 \leq x \leq \lambda q_1$ implies $|m(x)| \leq \lambda$ for all $m \in M$, hence $\|x\| \leq \inf\{\lambda > 0: -\lambda q_1 \leq x \leq \lambda q_1\}$. Conversely, since $|m(x)| \leq \|x\|$ for all $m \in M$, it follows that $-\|x\| q_1 \leq x \leq \|x\| q_1$, and (2.1) follows.

Moreover, $-q_1 \leq x \leq q_1$ implies $|m(x)| \leq 1$ for all $m \in M$, hence $sp(x) \subset [-1, 1]$. But then $sp(x^2) \subset [0, 1]$, that is, $0 \leq x^2 \leq q_1$. This proves that (2.2) holds.

Norm completeness follows by Proposition 2.5.

3. SEGAL PRODUCT

Let (L, M) be a u-spectral logic. Define the *Segal product on* $\mathbb{O}(L)$ by

$$x.y = \frac{1}{4}[(x + y)^2 - (x - y)^2] \tag{3.1}$$

The Segal product is always commutative, but not necessarily distributive. If observables x, y, z are mutually compatible, then by the functional calculus they can be considered respectively as Borel functions f, g, h of an observable u, and therefore the distributivity conditions $x.(y + z) = x.y + x.z$ is satisfied. Since the observables q_1 and q_0 are compatible with any observable x, it follows that $x.q_0 = q_0$, $x.q_1 = x$.

Lemma 3.1. (Hudson and Pulmannová, 1993). If the Segal product is distributive, it is (real) bilinear.

Proof. To sketch the proof, observe that by commutativity and distributivity it is sufficient to prove that, for arbitrary $x.\ y \in \mathbb{O}(L)$ and $r \in \mathbb{R}$,

$$(rx).y = r(x.y) \tag{3.2}$$

For any $m \in M$, $m(rx) = \int r\lambda m(x(d\lambda)) = rm(x)$. Applying it to a natural number n, we get $nx = x + \cdots + x$ (n times), and (3.2) then follows by distributivity. Replacing x by $(1/n)x$, it can be seen that (3.2) also holds if r is rational. Using the Schwarz inequality $m(u.v)^2 \leq m(u^2)m(v^2)$ and a sequence of rationals $\{q_n\}$ converging to a real number r, we obtain $m((rx).y) = \lim_n m((q_n x).y) = \lim_n m(q_n(x.y)) = m(r(x.y))$, and since $m \in M$ is arbitrary, we deduce that (3.2) is satisfied. ■

In what follows, we put into relation u-spectral logics with order unit spaces and JB-algebras.

We recall that a (real) *Jordan algebra* is a real vector space A equipped with a bilinear product $(a, b) \to a \circ b$ such that, for all a, $b \in A$, the following hold:

(i) $a \circ b = b \circ a$
(ii) $a \circ (b \circ a^2) = (a \circ b) \circ a^2$

A *JB-algebra* is a Jordan Algebra A over the reals with identity element 1 equipped with a complete norm satisfying the following requirements for a, $b \in A$:

(i) $\|a \circ b\| \leq \|a\|\|b\|$
(ii) $\|a^2\| = \|a\|^2$
(iii) $\|a^2\| \leq \|a^2 + b^2\|$

Relations between order unit spaces and JB-algebras are stated in the following theorem.

Theorem 3.2 (Alfsen *et al.*, 1978). If A is a JB-algebra, then the set A^2 of all squares in A is a proper cone organizing A to a (norm) complete order-unit space whose distinguished order unit is the multiplicative identity element and whose norm is the given one, and such that condition (2.1) holds.

Conversely, if A is a complete order-unit space equipped with a Jordan product for which the distinguished order unit acts as an identity element and such that requirement (2.2) is satisfied, then A is a JB-algebra in the order-unit norm (2.1).

Theorem 3.3. If the Segal product in the set $\mathcal{O}(L)$ of bounded real observables on a u-spectral logic (L, M) is distributive, then $\mathcal{O}(L)$ is a JB-algebra.

Proof. It is easy to see that conditions (i)–(iii) from the definition of a JB-algebra are satisfied, even without the distributivity assumption, Indeed, using the fact that $sp(x^2) = sp(x)^2$, (ii) follows by the second definition of the norm $\|x\|$. Similarly, (iii) follows at once from the first definition of the norm. To prove (i), we may assume without loss of generality that $\|x\| \leq 1$, $\|y\| \leq 1$. Using (ii) and the first definition of the norm, we get

$$\begin{aligned}\|x.y\| &= \left\|\frac{1}{4}(x+y)^2 - \frac{1}{4}(x-y)^2\right\| \\ &\leq \frac{1}{4}\max(\|x+y\|^2, \|x-y\|^2) \\ &\leq 1\end{aligned}$$

Owing to Theorem 3.2 and Theorem 2.8, it suffices to prove that distributive Segal product is Jordan product. It is bilinear by Lemma 3.1 and clearly commutative. In establishing the Jordan property $x.(y.x^2) = (x.y).x^2$, we follow the pattern of Hudson and Pulmannová (1993), Theorem 5.4. From $\|x.y\| \leq \|x\|\|y\|$ we see that multiplication is norm continuous. From the functional calculus for compatible observables it follows that simple observables are norm dense in $\mathcal{O}(L)$. Therefore we may assume that x is simple, that is, $x = \Sigma_{i=1}^{n} \lambda_i q_{a_i}$ for real numbers $\lambda_1, \ldots, \lambda_n$ and mutually orthogonal $a_1, \ldots, a_n \in L$. Then the Jordan property is equivalent to

$$\sum_{i,j=1}^{n} \lambda_i \lambda_j^2 (q_{a_i}.y).q_{a_j} = \sum_{i,j=1}^{n} \lambda_i \lambda_j^2 q_{a_j}.(y.q_{a_j})$$

and hence to

$$(u.y).v = u.(y.v) \tag{3.3}$$

for orthogonal $u, v \in L$. From the functional calculus for a single observable it is clear that the subalgebra generated by a single element is always associative, hence the $\mathcal{O}(L)$ is a power-associative algebra. It then follows from Lemma 5.2 in Schafer (1996) that (3.3) holds. Hence $\mathcal{O}(L)$ endowed with the Segal product is a Jordan algebra. ■

4. A COMPARISON WITH THE ALFSEN AND SHULTZ THEORY

We recall that in Alfsen and Schultz (1976) an order-unit space (A, e) and a base-norm space (V, K) in separating order and norm duality are considered. That is, there is a bilinear form $\langle , \rangle$: $A \times V \to \mathbb{R}$ such that the following conditions are satisfied:

$$\begin{aligned} a \geq 0 &\Leftrightarrow \langle a, v \rangle \geq 0 \text{ for all } v \geq 0 \\ v \geq 0 &\Leftrightarrow \langle a, v \rangle \geq 0 \text{ for all } a \geq 0 \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} \|a\| \leq 1 &\Leftrightarrow |\langle a, v \rangle| \leq 1 \qquad \text{whenever } \|v\| \leq 1 \\ \|v\| \leq 1 &\Leftrightarrow |\langle a, v \rangle| \leq 1 \qquad \text{whenever } \|a\| \leq 1 \end{aligned} \tag{4.2}$$

We shall use the terms "weak" and "weakly" to denote the weak topologies defined to A and V by this duality.

We shall be concerned with weakly continuous positive projections with norm at most one P: $X \to X$, where X is either A or V (here "projection" means any idempotent map). For such projections we define

$$\ker^+ P = (\ker P) \cap X^+, \qquad \mathrm{im}^+ P = (\mathrm{im} P) \cap X^+ \tag{4.3}$$

We shall denote by P^* the dual projection of P. The dual projection will also be of norm at most one in virtue of norm duality.

Two weakly continuous positive projections P, Q: $X \to X$ are said to be *quasicomplementary* if

$$\ker^+ P = \mathrm{im}^+ Q, \qquad \mathrm{im}^+ P = \ker^+ Q \tag{4.4}$$

We shall also say that Q is a *quasicomplement* of P and vice versa.

A projection P: $X \to X$ is said to be *smooth* (with respect to the given duality) if it is weakly continuous, positive, and satisfies the requirement

$$y \in Y^+, y = 0 \text{ on } \ker^+ P \Rightarrow y = 0 \text{ on } \ker P \tag{4.5}$$

According to Alfsen and Shultz (1976), a projection on either of the two spaces A or V which is smooth with norm at most 1 and admits a smooth quasicomplement with norm at most 1 is said to be a *P-projection*. The quasicomplement of a P-projection is unique, and we denote it by P'. Clearly P' is also a P-projectionn.

For a given P-projection on A, the element Pe will be in the order interval $[0, e]$, and such elements Pe will be called *projective units* of A. Moreover, the set $F_P = (\mathrm{im} P^*) \cap K$ will be a face of K, and such faces F_P will be called *projective faces* of K. The set of all P-projections on A will be denoted by $\mathcal{P}$, the set of all projective units in A by $\mathcal{U}$, and the set of all projective faces of K by $\mathcal{F}$.

The relations between P-projections, projective units, and projective faces can be seen explicitly from the following theorem (cf. Theorem 2.17 in Alfsen and Shultz, 1976).

Theorem 4.1. Let P be a P-projection on A with associated projective unit Pe and projective face $F_P = (\mathrm{im} P^*) \cap K$. Then

$$Pe = \inf\{b \in A\colon \chi_{F_p} \leq b \leq 1 \text{ on } K\} \tag{4.6}$$

and

$$\begin{aligned} F_P &= \{\rho \in K\colon \langle Pe, \rho\rangle = 1\} \\ F_P^{\#} &= \{\rho \in K\colon \langle Pe, \rho\rangle = 0\} \end{aligned} \tag{4.7}$$

For given $a \in A^+$

$$Pa = a \text{ on } F_P, \qquad Pa = 0 \text{ on } F_P^{\#} \tag{4.8}$$

and Pa is the unique element of A^+ with this property. Specifically, Pa is given by the formula

$$\begin{aligned} Pa &= \sup\{b \in A^+\colon b \leq a \text{ on } F_P,\, b = 0 \text{ on } F_P^{\#}\} \\ &= \inf\{b \in A^+\colon b \geq a \text{ on } F_P,\, b = 0 \text{ on } F_P^{\#}\} \end{aligned} \tag{4.9}$$

As a corollary, the sets $\mathcal{P}$, $\mathcal{U}$, $\mathcal{F}$ are canonically order isomorphic. Specifically, we have a commutative diagram of order-preserving maps where $\mathcal{P} \to \mathcal{U}$ is given by $P \to Pe$, $\mathcal{U} \to \mathcal{F}$ is given by the first equality of (4.7), and $\mathcal{F} \to \mathcal{P}$ is given by (4.9). Moreover, the operations $P \to P'$, $Pe \to e - Pe$, and $F \to F^{\#}$ will correspond to each other under these isomorphisms (Alfsen and Shultz, 1976, Corollary 2.18).

Recall that a face F of K is *exposed* if there is a weakly closed affine hyperplane H in V such that $F = H \cap K$. This means that there shall exist an $a \in A$ and an $\alpha \in \mathbb{R}$ such that

$$\langle a, \rho\rangle = \alpha \text{ for } \rho \in F, \qquad \langle a, \rho\rangle > \alpha \text{ for } \rho \in K \setminus F$$

Note that by (4.7) every projective face is exposed (Alfsen and Shultz, 1976, Proposition 2.15).

In accordance with Alfsen and Shultz (1976), we shall impose the following two requirements:

$$A \text{ is pointwise monotone } \sigma\text{-complete} \tag{4.10}$$

$$\text{Every exposed face of } K \text{ is projective} \tag{4.11}$$

The requirement (4.10) means that if $\{a_n\}$ is an increasing sequence from A which is bounded above, then there exists $a \in A$ such that $\langle a, \rho \rangle = \sup_n \langle a_n, \rho \rangle$ for all $\rho \in K$. In this case we shall write $a = \sup_n a_n$. By duality, the same statement holds for the pointwise infimum $\inf_n a_n$ of a descending sequence.

The requirement (4.11) is a strong one, but we will see later that it is implied by a spectral axiom.

The following result has been obtained in Alfsen and Shultz (1976).

Theorem 4.2 (Alfsen and Shultz, 1976, Theorem 4.5). If the conditions (4.10) and (4.11) are satisfied, then the set $\mathcal{P}$ is a α-complete orthomodular lattice, that is, for P and Q in $\mathcal{P}$:

(i) $P'' = P$
(ii) $P \leq Q \Rightarrow Q' \leq P'$
(iii) $P \leq Q \Rightarrow Q = P + (Q \wedge P')$
(iv) $(P_i)_i,\ P_i \leq P_j' \ (i \neq j) \Rightarrow \vee_i P_i \in \mathcal{P}$

Moreover, there is an analogue of the range projection in a von Neumann algebra. We use the notation face(a) to denote the smallest face of A^+ containing a given element a of A^+.

Proposition 4.3. For each $a \in A^+$ there exists a smallest projective unit h such that $a \in \text{face}(h)$, and h is the unique element of $\mathcal{U}$ such that for $\rho \in K$

$$\langle h, \rho \rangle = 0 \Leftrightarrow \langle a, \rho \rangle = 0 \tag{4.12}$$

Moreover, $a \leq \|a\| h$.

For given $a \in A^+$ we shall denote the projective unit h of Proposition 4.3 by rp(a).

The following definition of compatibility has been introduced (Alfsen and Shultz, 1976, p. 32): A P-projection P on A and an element a of A are said to be *compatible* if $Pa + P'a = a$. We also have the following characterization: P is compatible with a iff $Pa \leq a$.

The next proposition shows the relations between the compatiblity of a P-projection P and the projective unit associated with a P-projection Q.

Proposition 4.4 (Alfsen and Shultz, 1976, Proposition 5.2). Let P and Q be P-projections; then the following are equivalent:

(i) PQ is a P-projection
(ii) $PQ = P \wedge Q$, i.e., PQ is the g.l.b. of P and Q in $\mathcal{P}$
(iii) P is compatible with Qe
(iv) Q is compatible with Pe
(v) $QP = PQ$

Proposition 4.4 allows us to introduce the following definition: two projections P and Q are said to be *compatible* if they satisfy the equivalent conditions (i)–(v) of Proposition 4.4, and this notion of compatibility is also transferred from $\mathcal{P}$ to the sets $\mathcal{U}$ and $\mathcal{F}$ which are order isomorphic with $\mathcal{P}$. The following proposition shows that the notion of compatibility coincides with the usual compatibility in orthomodular lattices. [We recall that two P-projections are said to be *orthogonal* if $P \leq Q'$, and write $P \perp Q$ (Alfsen and Shultz, 1976, p. 28).]

Proposition 4.5. Two projections P and Q are compatible iff there exist mutually orthogonal P-projections, R, S, T such that

$$P = R + S, \qquad Q = S + T \tag{4.13}$$

If such a decomposition eixsts, it is unique; in fact

$$R = P \wedge Q', \qquad S = P \wedge Q, \qquad T = Q \wedge P'$$

To summarize the results obtained so far, under conditions (4.10) and (4.11), the set $\mathcal{U}$, which is a subset of A, is a σ-complete orthomodular lattice. Moreover, it can be shown that every state is Jauch–Piron. Indeed, let $a, b \in \mathcal{U}$; then there are P-projections P, Q such that $a = Pe$, $b = Qe$. If $m(a) = \langle Pe, m\rangle = 1$, $m(b) = \langle Qe, m\rangle = 1$, then by (4.7), $m \in F_P$, $m \in F_Q$, and $F_{a\wedge b} = F_a \cap F_b$ by Alfsen and Shultz (1976), Lemma 4.1. This entails $m(a \wedge b) = 1$. As a corollary, the statement of Lemma 2.7 holds (compare with Alfsen and Shultz, 1976, Proposition 8.7).

In the next step, we need to bring into relation the elements of A with bounded observables on $\mathcal{U}$. This is done by "spectral axioms" introduced in Alfsen and Shultz (1976). We note that condition (4.10), which says that A is pointwise monotone σ-complete, entails that A is norm complete (Alfsen and Shultz, 1976, Proposition 6.1).

Let (A, E) and (V, K) be order-unit and base-norm space, respectively, which are in separating order and norm duality. The spaces A and V will be said to be in *weak spectral duality* if A is monotone pointwise σ-complete and if for every $a \in A$ and $\lambda \in \mathbb{R}$ there exists a projective face F compatible with a such that

$$a \leq \lambda \text{ on } F,\ a > \lambda \text{ on } F^{\#} \tag{4.14}$$

By definition weak spectral duality implies (4.10), and by Alfsen and Shultz (1976), Proposition 6.2, it also implies (4.11), that is, if (A, e) and (V, K) are in weak spectral duality, then every exposed face of K is projective.

The following concept of orthogonality for elements of A^{+}, which is of interest in itself, enables us to reformulate the definition of weak spectral duality.

Let the conditions (4.10) and (4.11) be satisfied. Then two elements a, $b \in A^+$ are said to be *orthogonal*, in symbols $a \perp b$, if $\mathrm{rp}(a) \perp \mathrm{rp}(b)$.

Proposition 4.6 (Alfsen and Shultz, 1976, Proposition 6.3). Let the conditions (4.10) and (4.11) be satisfied; then A and V will be in a weak spectral duality iff every element $a \in A$ admits a decomposition $a = a_1 - a_2$ with $a_1, a_2 \in A^+$ and $a_1 \perp a_2$.

Assume that A and V are in weak spectral duality. A family $\{e_\lambda\}_{\lambda \in \mathbb{R}}$ of projective units is said to be a *spectral family* if for $\lambda, \mu \in \mathbb{R}$ we have the following:

(i) $e_\lambda \leq e_\mu$ when $\lambda \leq \mu$
(ii) $e_\lambda = \wedge_{\mu > \lambda} e_\mu$
(iii) $\wedge_{\lambda \in \mathbb{R}} e_\lambda = 0$, $\vee_{\lambda \in \mathbb{R}} e_\lambda = e$

We shall say that such a family has *compact support* if there exist $\alpha, \beta \in \mathbb{R}$ such that $e_\lambda = 0$ for all $\lambda \leq \alpha$ and $e_\lambda = e$ for all $\lambda \geq \beta$.

A spectral family $\{e_\lambda\}$ is said to be a *spectral resolution* for a given element $a \in A$ if for every $\lambda \in \mathbb{R}$ the projective face F_λ corresponding to e_λ is compatible with a and satisfies

$$a \leq \lambda \text{ on } F_\lambda, \qquad a > \lambda \text{ on } \mathrm{F}_\lambda^\# \tag{4.15}$$

We shall use the term *partition of* $[\alpha, \beta]$ to denote a finite sequence $\gamma = \{\lambda_i\}_{i=0}^n$ such that

$$\alpha = \lambda_0 < \lambda_1 < \ldots < \lambda_n = \beta$$

We shall use the symbol $\|\gamma\|$ to denote the *norm* of the partition, i.e., $\|\gamma\| = \max|\lambda_i - \lambda_{i-1}|$.

Theorem 4.7 (Alfsen and Shultz, 1976, Theorem 6.8 and Corollaries 6.9 and 6.10). Assume A and V are in weak spectral duality. Then each $a \in A$ will admit a spectral resolution, and if $\{e_\lambda\}$ is any spectral resolution of a, then

$$a = \int \lambda de_\lambda \tag{4.16}$$

where the right-hand side is a norm-convergent Riemann–Stiltjes integral. Moreover, $[-\|a\|, \|a\|]$ is the support of $\{e_\lambda\}$.

Conversely, to any spectral family $\{e_\lambda\}$ of compact support there exists a unique element $a \in A$ such that $\{e_\lambda\}$ is a spectral resolution of a and is given by (4.16).

From Theorm 4.7 it follows that if A and V are in weak spectral duality, then A can be considered as the space of all bounded observables on the σ-complete orthomodular lattice $\mathcal{U}$. Elements of K represent states on $\mathcal{U}$, which are σ-additive owing to the pointwise monotone σ-completeness of A. Moreover, the set K is rich for $\mathcal{U}$. Indeed, we have the following equivalence valid for projective units h, k:

$$h \leq k \Leftrightarrow \{\rho \in K: \langle k, \rho\rangle = 0\} \subset \{\rho \in K: \langle h, \rho\rangle = 0\}$$

[Alfsen and Shultz, 1976, (4.15)]. But the correspondence between observables and elements of A may be not one-to-one. Translated to the language of quantum logics, observables are not uniquely defined by their expectation values ($\langle a, \rho\rangle$, $a \in A$, $\rho \in K$), so that the uniqueness property of Gudder (1966) is not satisfied. To achieve this, we need to strengthen the condition of weak spectral duality.

According to Alfsen, and Shultz (1976), we say that A and V are in *spectral duality* if A is pointwise monotone σ-complete and if for every $a \in A$ and $\lambda \in \mathbb{R}$ there exists a projective face F which is bicompatible with a (in the sense that F is compatible with a and with all projective faces compatible with a) and satisfies

$$a \leq \lambda \text{ on } F, \qquad a > \lambda \text{ on } F^{\#} \tag{4.17}$$

Theorem 4.8 (Alfsen and Shultz, 1976, Theorem 7.2). If A and V are in spectral duality, then every $a \in A$ has a unique spectral resolution $\{e_\lambda\}$. For given $\lambda \in \mathbb{R}$ the projective face F_λ corresponding to e_λ is the unique member of $\mathcal{F}$ which is compatible with a and satisfies (4.17), and the P-projection P_λ corresponding to e_λ is the supremum of all $Q \in \mathcal{P}$ which are compatible with a and satisfy the inequality $Qa \leq \lambda Qe$.

We have the following relation between spectral duality and weak spectral duality: if A and V are in weak spectral duality and every $a \in A$ has a unique spectral resolution, then A and V are in spectral duality (Alfsen and Shultz, 1976, Theorem 7.5).

Moreover, If A and V are in spectral duality, then there is a one-to-one correspondence between elements $a \in A$ and spectral families $\{e_\lambda\}$ of compact support given by

$$a = \int \lambda de_\lambda$$

where the right side is a norm-convergent Riemann-Stiltjes integral (Alfsen and Shultz, 1976, Theorem 7.6). In other words, there is a one-to-one correspondence between elements of A and bounded observables on the orthomodular σ-lattice $\mathcal{U}$ of projective units (Alfsen and Shultz, 1976, Propositions 8.2 and 8.3), and there is a functional calculus (Alfsen and Shultz, 1976, Theorem 8.9) which coincides with that in Varadarajan (1985).

Now we wish to find conditions under which $\mathcal{U}$ can be considered as a u-spectral logic.

Let a: $K \to [0, 1]$ be a convex function. An arbitrary $\rho \in V$ can be written in the form $\rho = \lambda\sigma - \mu\tau$, where $\sigma, \tau \in K$ and $\lambda, \mu \in \mathbb{R}^+$. Now a can be uniquely extended to a bounded linear functional on V, which we also denote by a. In fact, for every $\rho \in V$, we obtain by linearity

$$\langle a, \rho\rangle = \lambda\langle a, \sigma\rangle - \mu\langle a, \tau\rangle$$

If $\lambda\sigma - \mu\tau = \lambda'\sigma' - \mu'\tau'$, where $\lambda, \lambda', \mu, \mu' \geq 0$ and $\sigma, \sigma', \tau, \tau' \in K$, then $\lambda\langle a, \sigma\rangle - \mu\langle a, \tau\rangle = \lambda'\langle a, \sigma'\rangle - \mu'\langle a, \gamma'\rangle$. To verify this, evaluate the linear functional e at both sides of $\lambda\sigma - \mu\tau = \lambda'\sigma' - \mu'\tau'$ to obtain $\lambda + \mu' = \lambda' + \mu$. We denote this common value by α. Then dividing by α, we obtain the quaility of two convex combinations

$$\lambda\alpha^{-1}\sigma + \mu'\alpha^{-1}\sigma' = \mu\alpha^{-1}\tau + \lambda'\alpha^{-1}\sigma$$

Since a preserves convex combinations, the desired result follows. Moreover,

$$|\langle a, \rho\rangle| = |\lambda\langle a, \sigma\rangle - \mu\langle a, \tau\rangle| \leq \lambda + \mu = \|\rho\|$$

which proves $\|a\| \leq 1$.

Let a be a bounded positive linear functional on V. Then $a/\|a\|$ restricted to K is a convex functional which maps K into $[0, 1]$. It follows that positive linear functionals on V are in one-one correspondence with convex functionals on K with values in $[0, 1]$. This yields the following result.

Theorem 4.9. Let (A, e) and (V, K) be an order unit and a base norm space in spectral duality. Then the set of projective units $(\mathcal{U}, K)$ is a u-spectral logic iff $A = V^*$.

Proof. Every convex functional a: $K \to [0, 1]$ extends to a positive bounded linear functional $\tilde{a}$ in V^*, hence $\tilde{a} \in A$ in case $V^* = A$.

Conversely, if every convex functional a: $K \to [0, 1]$ corresponds to an element in A, then since every positive bounded linear functional in V^* restricted to K is a convex functional from K to $[0, 1]$, and since V^* is generated by positive functionals, we get $V^* = A$. ■

5. U-SPECTRAL LOGICS WITH CONDITIONING

In this section we consider a u-spectral logic (L, M). We shall find conditions under which the corresponding order-unit and base-norm spaces are in spectral duality.

After Pool (1968a, b), and Guz (1980), let us define to each $a \in L$ a mapping E_a of the set of states M into itself whose domain is the set $D(E_a) := \{m \in M\colon m(a) > 0\}$ such that[2]

(C1) $(E_a m)(a) = 1$ for all $m \in D(E_a)$
(C2) $E_a m = m$ whenever $m(a) = 1$

The following properties of E_a are easily verified:

(i) $0 < a \le b \Rightarrow E_b E_a m = E_a m$ for all $m \in D(E_a)$. Indeed, $E_a m(a) = 1$ by (C1), and hence by (C2), $E_b E_a m = E_a m$.
(ii) $a \perp b$ $(a, b \in L \setminus \{0\})$ implies $E_a m(b) = 0$ for each $m \in D(E_a)$. Indeed, $E_a m(a') = 0$ by (C1) and $b \le a'$.

It will be convenient to pass from E_a to P_a: $V_+ \to V_+$ defined by

$$P_a x = \langle a, x\rangle\, E_a\left(\frac{x}{\|x\|}\right) \qquad \text{when} \quad \langle a, x\rangle > 0$$

$$P_a x = 0 \qquad \text{when } \langle a, x\rangle = 0$$

P_a is defined also for $a = 0$, and clearly $P_0 = 0$. The following properties of P_a are easily verified:

(i) $\|P_a x\| = \langle a, x\rangle = \langle a, P_a x\rangle$
(ii) For every $x \in V_+$ and $t \ge 0$, $P_a(tx) = tP_a x$
(iii) $\|P_a x\| \le \|x\|$, and $\|P_a x\| = \|x\|$ iff $P_a x = x$
(iv) $a \le b$ $(a, b \in L)$ implies $P_b P_a = P_a$; in particular, $P_a^2 = P_a$
(v) $a \perp b$ $(a, b \in L)$ implies $P_b P_a = 0$

Indeed, (i) follows directly from the fact that $\|x\| = \langle e, x\rangle$.
To prove (ii), observe that

$$\begin{aligned} P_a(tx) &= \langle a, tx\rangle E_a\left(\frac{tx}{\|tx\|}\right) \\ &= \langle a, tx\rangle E_a\left(\frac{x}{\|x\|}\right) \\ &= t\langle a, x\rangle E_a\left(\frac{x}{\|x\|}\right) = tP_a x \end{aligned}$$

[2] The mapping E_a has a straightforward physical interpretation. If, after a measurement performed on a physical system that is initially in the state m, and proposition a is verified to be true, then the sequential state of the system is $E_a m$.

The first part of (iii) follows by (i) since $\|P_a x\| = \langle a, x\rangle \leq \langle e, x\rangle$. Assume that $\|P_a x\| = \|x\|$ for some $x \in V_+$. If $\langle a, x\rangle \neq 0$, we have $\langle a, x\rangle = \|x\|$, hence $\langle a, x/\|x\|\rangle = 1$, which by (C2) implies that $E_a(x/\|x\|) = x/\|x\|$. If $\langle a, x\rangle = 0$, we get from $\langle a, x\rangle = \langle e, x\rangle$ that $x = 0 = P_a x$. This proves the second part of (iii).

If $a \leq b$, $a, b \in L$, we get for any $x \in V_+$ that $\|P_b P_a x\| = \langle b, P_a x\rangle \geq \langle a, P_a x\rangle = \|P_a x\|$. The inverse inequality follows by (iii). Hence $\|P_b P_a x\| = \|P_a x\|$, and by the second part of (iii), $P_b P_a x = P_a x$. This proves (iv).

If $a \perp b$ $a, b \in L$, by (i) we have

$$\begin{aligned} \|P_a P_b x\| &= \langle a, P_b x\rangle \leq \langle b', P_b x\rangle \\ &= \langle e, P_b x\rangle - \langle b, P_b x\rangle = 0 \end{aligned}$$

so that $P_a P_b x = 0$, hence (v) is proved.

Lemma 5.1. The map P_a: $m \mapsto P_a m$ is affine, i.e., for any $m_1, m_2 \in M$ and $t \in [0, 1]$.

$$P_a(tm_1 + (1 - t)m_2) = tP_a m_1 + (1 - t)P_a m_2$$

Proof. Since the norm $\|\cdot\|$ is additive on V^+, we have

$$\begin{aligned} \|tP_a(m_1) + (1 - t)P_a(m_2)\| &= \|tP_a(m_1)\| + \|(1 - t)p_a(m_2)\| \\ &= t\|P_a(m_1)\| + (1 - t)\|P_a(m_2)\| \\ &= t\langle a, m_1\rangle + (1 - t)\,\langle a, m_2\rangle \\ &= \langle a, tm_1 + (1 - t)m_2\rangle \\ &= \|P_a(tm_1 + (1 - t)m_2\| \end{aligned}$$

and the result follows by (iii). ■

Obviously, P_a may be uniquely extended to a linear mapping acting on the whole space V. It will be also denoted by P_a, as this does not lead to a misunderstanding.

Proposition 5.2. For each $a \in L$ and each bounded observable $A \in \mathbb{O}^+(L)$, there is a bounded observable $B \in \mathbb{O}^+(L)$ such that $\langle B, m\rangle = \langle A, P_a(m)\rangle$ for all $m \in M$.

Proof. Assume $0 < A \leq I$. Then $m \to \langle A, P_a m\rangle$ is a convex mapping from M to $[0, 1]$, hence there is an observable $B \in \mathbb{O}(L)$ such that $\langle B, m\rangle = \langle A, P_a m\rangle$ for each $m \in M$. If $A \neq 0$, consider $0 < A/\|A\| \leq I$ and put $B = \|A\|B_1$, where B_1 is such that $\langle B_1, m\rangle = \langle A/\|A\|, P_a m\rangle$ for each $m \in M$. If $A = 0$, clearly, $\langle A, P_a m\rangle = 0$ for each m, hence $B = 0$. ■

Obviously, B is unique and we denote it by $Q_a A$. It is easy to show that the mapping $A \mapsto Q_a A$ is positive homogeneous. It can be extended to a linear positive mapping at the whole $\mathbb{O}(L)$ into itself. We shall call P_a a filter and Q_a a dual filter.

Lemma 5.3. For any $a \in L$, the mapping P_a: $V \to V$ is continuous with respect the weak topology in V given by the duality $\langle .,. \rangle$.

Proof. Let $\{m_\alpha\}$ be a net such that $m_a \xrightarrow{w} m$, i.e., $\langle x, m_\alpha \rangle \to \langle x, m \rangle$ for all $x \in \mathbb{O}(L)$. By Proposition 5.2, there is a positive linear mapping Q_a: $\mathbb{O}(L) \to \mathbb{O}(L)$ such that $\langle x, P_a m_\alpha \rangle = \langle Q_a x, m_\alpha \rangle \to \langle Q_a x, m \rangle = \langle x, P_a \mathrm{m} \rangle$, i.e., $P_a m_\alpha \xrightarrow{w} P_a m$. ■

We say that two filters P_a, P_b are *compatible* and write $P_a \leftrightarrow P_b$ if for each state $m \in M$,

$$\|P_b(P_a + P_{a'})m\| = \|P_b m\|, \qquad \|P_a(P_b + P_{b'})m\| = \|P_a m\|$$

From the definition it follows easily that

$P_a \leftrightarrow P_b$ iff $P_b \leftrightarrow P_a$
$P_a \leftrightarrow P_b$ implies $P_{a'} \leftrightarrow P_b$
$P_a \leftrightarrow P_b$ implies $P_a \leftrightarrow P_{b'}$

Passing to dual filters, we define $Q_a \leftrightarrow Q_b$ iff $P_a \leftrightarrow P_b$. Moreover, we define compatibility of a dual filter Q_a with $b \in L$ by $Q_a \leftrightarrow b$ iff $Q_a \leftrightarrow Q_b$, and if $A \in \mathbb{O}(L)$, define $Q_a \leftrightarrow A$ iff $Q_a A + Q_{a'} A = A$.

It is not difficult to see that for every $a \in L$, P_a and Q_a are positive projections (i.e., idempotent linear mappings). Recall that a weakly continuous and positive projection P on V is said to be *neutral* if it is of norm at most one and $\|P\rho\| = \|\rho\| \Rightarrow \rho \in \mathrm{im}^+ P$ whenever $\rho \in V^+$. We see from the properties (i)–(iii) of P_a: $V^+ \to V^+$ that every P_a is neutral.

Lemma 5.4. If $Q_a \leftrightarrow Q_{A(E)}$ for all $E \in \mathcal{B}(\mathbb{R})$, then $Q_a \leftrightarrow A$.

Proof. From $Q_a \leftrightarrow Q_{A(E)}$ for all $E \in \mathcal{B}(\mathbb{R})$ we have, for any $x \in \mathrm{V}^+$,

$$\begin{aligned}\|P_{A(E)} P_a x\| + \|P_{A(E)}\, P_{a'} x\| &= \|P_{A(E)}(P_a + P_{a'})x\| \\ &= \|P_{A(E)} x\|\end{aligned}$$

Then for all $x \in M$,

$$\begin{aligned}\langle Q_a A + Q_{a'} A, x \rangle &= \int_{-\infty}^{\infty} t P_a x(A(dt)) + \int_{-\infty}^{\infty} t P_{a'} x(A(dt)) \\ &= \int_{-\infty}^{\infty} t\|P_{A(dt)} P_a x\| + \int_{-\infty}^{\infty} t\|P_{A(dt)} P_{a'} x\|\end{aligned}$$

$$= \int_{-\infty}^{\infty} t \left\| P_{A(dt)} x \right\|$$

$$= \int_{-\infty}^{\infty} tx(A(dt)) = \langle A, x \rangle$$

which leads to $Q_a A + Q_{a'} A = A$. ■

Proposition 5.5. Let (L, M) be a u-spectral logic such that a mapping $a \to E_a$ satisfying (C1) and (C2) is defined on L. The following statements are equivalent $[A \in \mathcal{O}(L),\ a, b \in L]$:

(i) If $A \geq 0$, then $Q_a \leftrightarrow A$ if and only if $Q_a A \leq A$.
(ii) Every P_a is a P-projection and so is Q_a.
(iii) For given $A \in \mathcal{O}^+(L)$ the observables, $B = Q_a A$ and $C = Q_{a'} A$ are the only elements of $\mathcal{O}^+(L)$ such that:
(1) $B = A$ on $\ker^+ P_{a'}$, $B = 0$ on $\mathrm{im}^+ P_{a'}$
(2) $C = A$ on $\ker^+ P_a$, $C = 0$ on $\mathrm{im}^+ P_a$

Proof. (i) $\Rightarrow$ (ii): We shall show that P_a and $P_{a'}$ are quasicomplementary, i.e., $\mathrm{im}^+ P_a = \ker^+ P_{a'}$ and $\mathrm{im}^+ Q_a = \ker^+ Q_{a'}$. By the symmetry reason we shall then have $\mathrm{im}^+ P_{a'} = \ker^+ P_a$ and $\mathrm{im}^+ Q_{a'} = \ker^+ Q_a$

Let $x \in \mathrm{im}^+ P_a$; then $x = P_a x$, so that $P_{a'} x = P_{a'} P_a x = 0$. Thus we have shown that $\mathrm{im}^+ P_a \subseteq \ker^+ P_{a'}$. Now let us assume that $x \in \ker^+ P_{a'}$. Then we get

$$\|x\| = \langle e, x \rangle = \langle a, x \rangle + \langle a', x \rangle$$
$$= \|P_a x\| + \|P_{a'} x\| = \|P_a x\|$$

and neutrality of P_a implies that $P_a x = x$, hence $x \in \mathrm{im}^+ P_a$. This show that $\ker^+ P_{a'} \subseteq \mathrm{im}^+ P_a$, and this completes the proof that $\mathrm{im}^+ P_a = \ker^+ P_{a'}$.

Now we shall pass to the proof that $\mathrm{im}^+ Q_a = \ker^+ Q_{a'}$. Assume that $A \in \mathrm{im}^+ Q_a$. Then $Q_a A = A$, so that for arbitrary $x \in V^+$, we have $\langle A, x \rangle = \langle A, P_a x \rangle$, hence for any state $m \in M$, $\langle A, P_{a'} m \rangle = \langle A, P_a P_{a'} m \rangle = 0$ or, equivalently, $\langle Q_{a'} A, m \rangle = 0$. This equality, valid for any $m \in M$, implies $Q_{a'} A = 0$, hence $A \in \ker^+ Q_{a'}$. This proves that $\mathrm{im}^+ Q_a \subseteq \ker^+ Q_{a'}$. To prove the converse inclusion, assume that $A \in \ker^+ Q_{a'}$. Then $Q_{a'} A = 0 < A$, so that $Q_{a'} \leftrightarrow A$, and hence

$$A = Q_a A + Q_{a'} A = Q_a A$$

i.e., $A \in \mathrm{im}^+ Q_a$. Thus $\ker^+ Q_{a'} = \mathrm{im}^+ Q_a$.

(ii) $\Leftrightarrow$ (iii): Taking into account that every P_a is neutral, the equivalence follows by Alfsen and Shultz (1976), Theorem 2.6.

(iii) $\Rightarrow$ (i): Follows by Alfsen and Shultz (1976), Proposition 5.1: If Q_a and A $(A \geq 0)$ are compatible, then $A = Q_a A + Q_{a'} A \geq Q_a A$. Conversely, if $Q_a A \leq A$, then $A - Q_a \mathrm{A} \geq 0$, and $Q_a(A - Q_a A) = Q_a A - Q_a A = 0$. Thus $A - Q_a A = Q_{a'}(A - Q_a A) = Q_{a'} A$, and so $A = Q_a A + Q_{a'} A$. ■

Theorem 5.6. If one of the equivalent conditions (i)–(iii) of Proposition 5.5 is satisfied, then V and $\mathbb{O}(L)$ are in weak spectral duality, i.e., for every $A \in \mathbb{O}(L)$ every $\lambda \in \mathbb{R}$, there is a projective face compatible with A such that

$$A \leq \lambda \text{ on } F, \qquad a > \lambda \text{ on } F^{\#}$$

Moreover, V and $\mathbb{O}(L)$ are in fact in spectral duality.

Proof. It suffices to prove the statement for $\lambda = 0$. For $\lambda \neq 0$, we consider $A - \lambda I$ instead of A.

Observe that for any $\Delta \in \mathcal{B}(\mathbb{R})$ we have, using properties (iv) and (v) of P_a,

$$\begin{aligned} Q_{A(\Delta)}\, A &= \mathrm{Q}_{A(\Delta)} \int tA(dt) \\ &= \int_{\Delta} tQ_{A(\Delta)}\, A(dt) + \int_{\Delta'} tQ_{A(\Delta)}\, A(dt) \\ &= \int_{\Delta} tA(dt) \end{aligned}$$

hence $Q_{A(\Delta)}\, A \leq A$ and by the assumptions, $Q_{A(\Delta)} \leftrightarrow A$.

Exchanging A by $-A$ if necessary, we may assume that $(spA) \cap \mathbb{R}^{+} \neq \{0\}$ and define

$$s = \inf\{t > 0\colon t \in spA\} \geq 0$$

Two cases may occur.

1. $s > 0$. In this case we have $(0, s) \cap spA = \emptyset$, so that for $m \in F$, where F is the projective face corresponding to $P_{A(-\infty, s)}$, we get $m(A(-\infty, s)) = 1$, so that

$$\langle A, m\rangle = \int_{spA} t\,\langle A(dt), m\rangle \leq 0$$

On the other hand, for $m \in F^{\#}$, $F^{\#}$ being the projective face associated with $P_{A[s,\infty)}$, we obtain

$$\langle A, m\rangle = \int_{spA} t\,\langle A(dt), m\rangle \geq s \geq 0$$

so that our statement is proved.

2. $s = 0$. Let $F = \text{im}^+ P_{A(-\infty,0)}$, as before. Let $\{t_n\}$, $t_n \downarrow 0$, $t_n \geq 0$, $t_n \in spA$, and let $R_1 = Q_{A[t_1,\infty)}$, $R_n = Q_{A[t_n,t_{n-1})}$, $n = 2, 3, \ldots$. Since for each i we have

$$R_1 \dot{+} R_2 \dot{+} \cdots \dot{+} R_i = Q_{A[t_i,\infty)} \leftrightarrow A$$

and $A \geq 0$ on F_i, the projective face associated with $R_1 \dot{+} R_2 \dot{+} \cdots \dot{+} R_i$, we get that $A > 0$ on F_0, the projective face corresponding to $R_1 \dot{+} R_2 \dot{+} \cdots \dot{+} R_i = Q_{A[0,\infty)}$, and the latter shows that $F_0 = F^{\#}$. Obviously, on F we have $a \leq 0$.

The last statement follows by the fact that every $a \in A$ has a unique spectral resolution with respect to elements in L, which by Lemma 2.7 (see also the remark after Theorem 4.7) coincides with extreme points $\mathcal{O}_1(L)$, and hence with the projective units. By Alfsen and Shultz (1976), Theorem 7.5, if A and V are in weak spectral duality and every $a \in A$ has a unique spectral resolution, then A and V are in spectral duality. ■

If the conditions of Theorem 5.6 are satisfied, then we have the following relation with Alfsen and Shultz theory: A corresponds to the set $\mathcal{O}(L)$ of all bounded observables on L, V is the ordered linear space induced by the convex cone base M. In addition, A and V are in spectral duality, V coincides with the set of all bounded linear functionals on $\mathcal{O}(L)$, the set of order units $\mathcal{U}$ corresponds to L, the projective faces on M correspond to the sets $\{m \in M: m(a) = 1\}$ for $a \in L$, and P-projections are the mappings Q_a, $a \in L$.

Conversely, let (A, e) and (V, K) be in order and norm duality. Let the conditions (4.10) and (4.11) be satisfied. Then the set $L \equiv \mathcal{U}$ of all projective units of A is a σ-complete orthomodular lattice (Alfsen and Shultz, 1976, Propositon 4.2). For every $x \in K$, denote by μ_x the state of L defined by

$$\mu_x(a) = \langle a, x \rangle$$

Assume that $\mu_x(a) \neq 0$ and define, for any $a \in L$,

$$E_a x := \frac{Px}{\|Px\|}$$

where P is the P-projection on V such that $P^*e = a$. Is is easy to see that conditions (C1) and (C2) are satisfied. Indeed, (C1):

$$\mu_{E_a x}(a) = \langle a, E_a x \rangle = \left\langle a, \frac{Px}{\|Px\|} \right\rangle$$

$$= \left\langle P^* e, \frac{Px}{\|Px\|} \right\rangle = \left\langle e, \frac{P^2 x}{\|Px\|} \right\rangle$$

$$= \left\langle e, \frac{Px}{\|Px\|} \right\rangle = \frac{\|Px\|}{\|Px\|} = 1$$

(C2): If $\mu_x(a) = 1$, then

$$\langle a, x\rangle = \langle P^*e, x\rangle = \langle e, Px\rangle = \|Px\| = 1$$

and hence $x \in im^+P$ by (4.7), i.e., $Px = x$, whence $E_a x = x$.

If $b \in L$ is compatible with a, and $b = Qe$ (Q is a P-projection in A), then

$$\mu_{E_a x}(b) = \langle b, E_a x\rangle = \left\langle Qe, \frac{Px}{\|Px\|}\right\rangle$$
$$= \left\langle P^*Qe, \frac{x}{\|Px\|}\right\rangle = \left\langle P^* \wedge Qe, \frac{x}{\|Px\|}\right\rangle$$
$$= \left\langle P^*e \wedge Qe, \frac{x}{\|Px\|}\right\rangle = \frac{\mu_x(a \wedge b)}{\mu_x(a)}$$

Using similar arguments as in Edwards and Rüttimann (1990), it can be shown that

$$\nu = \mu_{Px/\|Px\|}$$

is the unique element in $\Delta := \{\mu_x :\in K\}$ such that

$$\nu(b) = \frac{\mu_x(a \wedge b)}{\mu_x(a)}$$

for any $b \in L$ compatible with a.

So for any $a \in L$, Q_a is a P-projection such that $Q_a e = a$, and $Q_a \leftrightarrow b$ iff $Q_a b \leq b$ means that $a \leftrightarrow b$ iff $Q_a b = a \wedge b$.

As a conclusion, we obtain the following result:

Theorem 5.7. Let (L, M) be a u-spectral logic. Then $(\mathbb{O}(L), q_1)$ and (V, M) are order unit and base norm space in spectral duality if and only if there is a mapping $a \mapsto E_a$ satisfying (C1), (C2), and one of the equivalent conditions of Proposition 5.5.

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